

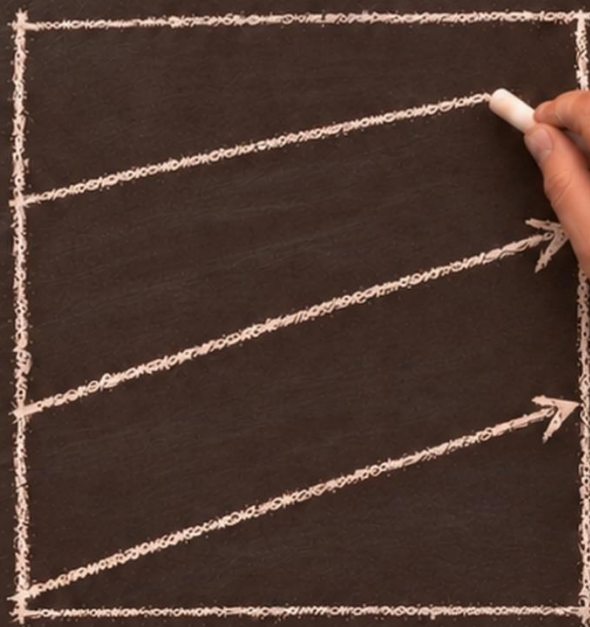


Corridor of Comprehensive Knowledge

MARYAM

COUNTING CLOSED PATHS

A MATHEMATICAL EXPLORATION IN HER MEMORY



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“Aha!” was an expressive word often heard from a brilliant mathematician at the exciting moment of discovery and new understanding. It was as if, after a long journey, she had reached the top of a hill and a clear landscape had suddenly opened before her [1].

Maryam Mirzakhani was born in Tehran in 1977. After studying at Sharif University of Technology, she received her Ph.D. in mathematics from Harvard University in 2004 and later became a professor at Stanford University. Her research, from counting simple closed paths on curved surfaces to studying the geometry and dynamics of complex spaces, created deep connections among several branches of mathematics [2].

In 2014, she received the Fields Medal for her outstanding contributions to “the dynamics and geometry of Riemann surfaces and their moduli spaces.” She was the first woman and the first Iranian to receive the medal [3].

Maryam Mirzakhani died on July 14, 2017, at the age of forty, after living with cancer. The problems, methods, and perspectives she introduced remain active in mathematics [4]. Her birthday, May 12, was chosen in her memory as a day to celebrate women in mathematics and has been observed around the world each year since 2019 [5].

From Visible Lattice Points to Counting Simple Paths

A Mathematical Exploration in Memory of Maryam Mirzakhani

Abstract

This article presents a continuous mathematical exploration for high-school students and readers interested in mathematics. It begins with a problem about motion at a fixed slope on a unit square and the formation of closed paths. To answer the questions raised by the problem, the path is unfolded onto an extended plane. This reveals its connection with rational slopes, coprime pairs, and lattice points. The visibility of a lattice point from the origin provides a geometric test for coprimality, while a bound on path length turns the problem into counting coprime points inside a quarter-disk. We first count all lattice points and then count the coprime ones. Using the inclusion-exclusion principle and the Möbius function, we obtain an asymptotic growth law for the number of paths. We also briefly examine the rectangular version of the problem. Finally, we compare the questions and results of this elementary model with Maryam Mirzakhani's work on counting simple closed curves. The model is neither a direct special case of Mirzakhani's theorem nor a historical reconstruction of how she reached it. It is an educational example of how a geometric question can become a counting problem and how mathematical work can move from observation and conjecture to proof.

Keywords: lattice points, coprime integers, simple closed path, asymptotic counting, Möbius function, Maryam Mirzakhani

1) Introduction

Maryam Mirzakhani was an outstanding Iranian mathematician and the 2014 Fields Medalist. Her research concerned the geometry and dynamics of surfaces and their parameter spaces. Part of her work addressed the number of simple closed curves whose lengths do not exceed a given bound. Although this may look like a counting question, its solution requires a deep combination of geometry, topology, measure theory, and dynamics [6]–[8].

The use of a flat two-dimensional model as an introduction to this subject also appears in reliable sources. An educational article published by the Oberwolfach Research Institute identifies simple closed paths on a square torus with primitive lattice vectors. Their quadratic growth law then provides an introduction to Mirzakhani's theorem on counting simple closed curves on hyperbolic surfaces [9]. In a research survey, Arana-Herrera likewise presents the counting of primitive lattice points as an elementary guide to the general mechanisms in Mirzakhani's proof [10]. The present article follows this educational approach, with emphasis on the first quadrant, a geometric estimate for lattice points, and an explicit use of the Möbius function.

We therefore study a very simple educational model whose general form resembles one aspect of Mirzakhani's much more difficult research. A length bound, the counting of distinct paths, and the search for a limiting growth law are common to both the elementary problem and her advanced work. The scale of Mirzakhani's achievement lies in solving a question of this kind on far more complicated surfaces and connecting the answer to the global structure of the space of surfaces.

The purpose of this article, however, is not to present a simplified version of Mirzakhani's theorems. The problem studied below belongs to a flat and highly regular geometry. It can be analyzed with number theory and high-school geometry, together with a short, more advanced supplementary section. Its educational value lies in showing a process of mathematical investigation: a problem that first appears complicated can become manageable after an appropriate transformation. The final result also has an interesting formal resemblance to the result obtained in Mirzakhani's research.

2) Statement of the Problem

Consider the unit square in the coordinate plane, with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Starting at $(0,0)$, move with a positive slope m . Whenever the path reaches one side of the square, continue from the corresponding point on the opposite side with the same slope. One may imagine folding the square so that corresponding points lie on top of each other. When the path reaches $(1,1)$, which corresponds to $(0,0)$, the path closes and the motion ends (Figure 1).

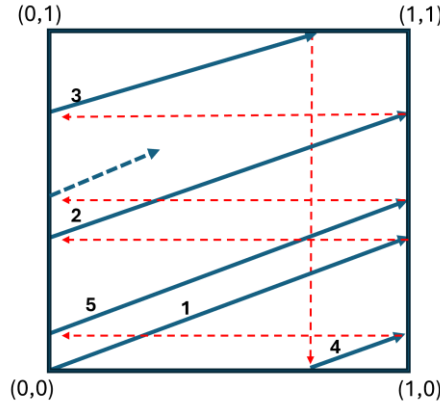


Figure 1) A path from $(0,0)$ to $(1,1)$.

We seek answers to the following questions.

Question 1: For which values of m does the path reach $(1,1)$ and therefore close? For which values does it fail to reach this point?

Question 2: Define the length of a path as the sum of the lengths of its segments inside the square. For a given number L , how many paths have length at most L ?

Question 3: How does the number of paths in Question 2 grow as L becomes large?

The following sections develop the ideas needed to answer these questions.

3) Conventions and Notation

The lattice plane consists of all points whose two coordinates are integers:

$$\mathbb{Z}^2 = \{(p, q) : p, q \in \mathbb{Z}\}.$$

The greatest common divisor of two positive integers p and q is denoted by $\gcd(p, q)$. The integers p and q are coprime when $\gcd(p, q) = 1$.

The notation $f(L) \sim g(L)$ means that $\lim_{L \rightarrow \infty} \frac{f(L)}{g(L)} = 1$.

The notation $E(L) = O(h(L))$ means that there is a constant $\delta > 0$, independent of L , such that for all sufficiently large L , $|E(L)| \leq \delta|h(L)|$.

Now consider the lattice point $(2,4)$ in Fig. 2. The point $(1,2)$ lies on the same line segment and occurs before $(2,4)$. Therefore, $(2,4)$ is not directly visible from the origin. In contrast, there is no other lattice point on the segment joining the origin to $(2,3)$.

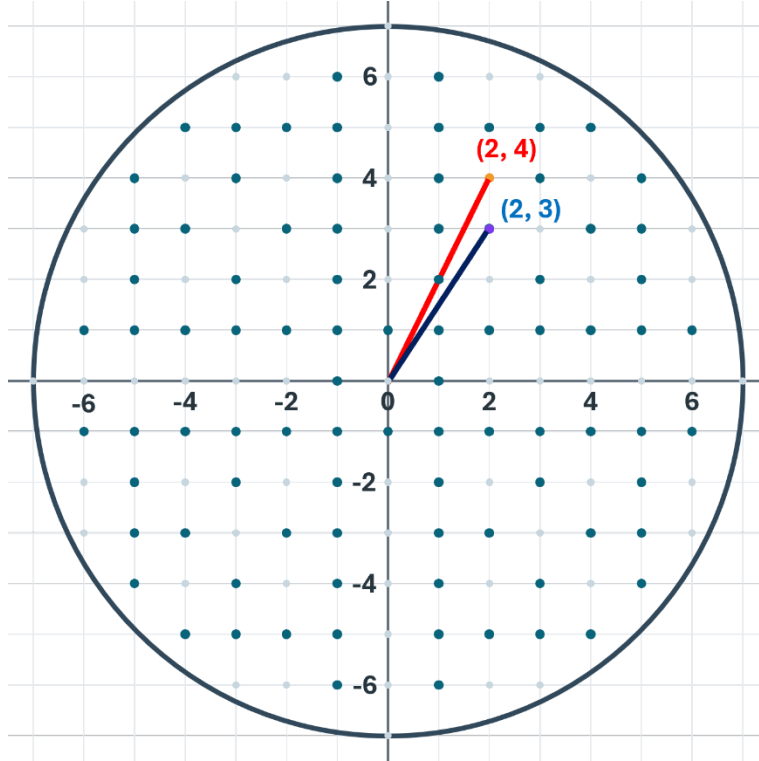


Figure 2) The dark points are visible from the origin.

We now show that a lattice point (p, q) is visible from the origin exactly when its coordinates are coprime. First assume that $\gcd(p, q) = d > 1$.

Then there are integers a and b such that $p = da, q = db$.

Consequently, $(p, q) = d(a, b)$.

The point (a, b) lies on the same ray from the origin as (p, q) . Because $d > 1$, it occurs before (p, q) . Thus, (p, q) is not visible from the origin.

Conversely, suppose that (p, q) is not visible. Among the finite, nonempty set of lattice points on the segment from the origin to (p, q) , let (a, b) be the first point after the origin. Its coordinates must be coprime. Otherwise, dividing both coordinates by a common divisor would produce another lattice point closer to the origin.

Since (a, b) and (p, q) lie on the same ray, there is a rational number $d > 1$ such that $(p, q) = d(a, b)$.

Write $d = r/s$ in lowest terms. From $p = ra/s, q = rb/s$, and the fact that p and q are integers, it follows that s divides both a and b . Since $\gcd(a, b) = 1$, we must have $s = 1$. Thus, d is an integer greater than one and is a common divisor of p and q . Therefore, (p, q) is not a coprime pair.

We have proved that (p, q) is visible from the origin exactly when $\gcd(p, q) = 1$.

Points satisfying this condition are also called primitive lattice points.

4) Solution of the Problem

We now return to the original problem. Suppose that the selected slope is rational:

$$m = \frac{q}{p},$$

where (p, q) is a coprime pair. If the path segments in Fig. 1 are placed end to end, beginning at $(0,0)$, the motion becomes a straight path from the origin to (p, q) . Intuitively, this unfolding can be produced by copying the unit square and rolling the copies horizontally and vertically along the path (Figure 3).

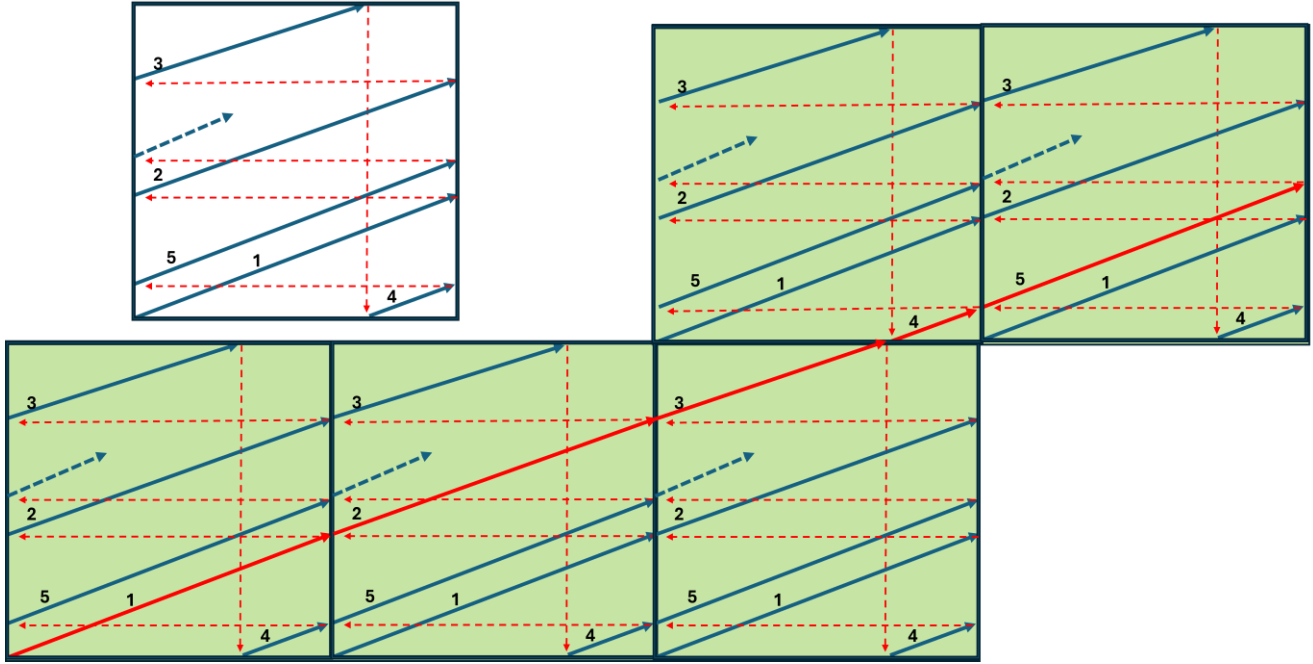


Figure 4) Unfolding the path into a straight line.

This construction ensures that the endpoints of all segments have rational coordinates and that the final point is (p, q) . The length of the path is $\sqrt{p^2 + q^2}$.

Thus, a closed path in the square with slope $m = q/p$ is equivalent to a straight segment in the coordinate plane from the origin to (p, q) . The term *closed* is used because $(1,1)$ corresponds to $(0,0)$ under the identification of opposite sides.

If (p, q) is not a coprime pair, then $(p, q) = d(a, b)$, where d is an integer and (a, b) is coprime. The path on the unit square repeats d times. It reaches $(1,1)$, and therefore returns to the corresponding point $(0,0)$, a total of d times. In the unfolded plane, the segment meets d lattice points, the last of which is (p, q) . When $d = 1$, or equivalently when (p, q) is coprime, we call the path simple.

We have shown that every rational slope $m = q/p$ produces a closed path on the square. If m is irrational, the path cannot close. To prove this, write the position in the unfolded plane as $(x(t), y(t)) = (t, mt)$.

For the corresponding path on the square to return to the starting point, or equivalently to reach $(1,1)$, both coordinates must become integers. Thus, there must be integers p and q such that $t = p$ and $mt = q$.

These equations imply that $m = \frac{q}{p}$.

Therefore, the path reaches (1,1) only if the slope is rational. This answers Question 1.

The length of the closed path, obtained by adding the lengths of its segments, is $\sqrt{p^2 + q^2}$. Therefore, counting simple paths of length at most L is equivalent to finding all coprime pairs (p, q) satisfying $\sqrt{p^2 + q^2} \leq L$.

The pairs corresponding to paths shorter than or equal to L lie inside the quarter-disk centered at the origin with radius L . This answers Question 2.

We now address Question 3 and study how the count grows as L increases.

Let $P(L)$ denote the number of positive integer pairs (p, q) satisfying both $p^2 + q^2 \leq L^2$ and $\gcd(p, q) = 1$.

The geometric path-counting problem has now become a precise problem in number theory.

Before counting coprime points, we count all lattice points in the first quadrant. Let $A(L)$ be the number of positive integer pairs (p, q) with $p, q > 0$ satisfying $p^2 + q^2 \leq L^2$.

Thus, $A(L)$ counts the lattice points in the first-quadrant quarter-disk centered at the origin with radius L .

To estimate $A(L)$, we first estimate the number of lattice points in the full disk centered at the origin with radius L . Around every lattice point, draw a unit square centered at that point, with sides parallel to the coordinate axes. Let $B(L)$ denote the total area of the squares whose centers lie inside the disk. Because every square has area one, $B(L)$ is numerically equal to the number of lattice points in the disk. The greatest distance from

the center of a unit square to one of its points is $c = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$.

It follows that $\pi(L - c)^2 \leq B(L) \leq \pi(L + c)^2$.

Expanding the two sides gives $\pi L^2 - 2\pi cL + \pi c^2 \leq B(L) \leq \pi L^2 + 2\pi cL + \pi c^2$.

Therefore, the difference between $B(L)$ and πL^2 is bounded by a quantity proportional to L . We write this result as $B(L) = \pi L^2 + O(L)$.

Here, $O(L)$ is not a particular number. It represents an error term whose magnitude grows at most proportionally to L . More precisely, there is a constant $\delta > 0$ such that, for all sufficiently large L :

$$|B(L) - \pi L^2| \leq \delta L. \quad (1)$$

The leading term in the lattice-point count is therefore the area of the disk, πL^2 , while the error grows at most proportionally to the radius. Geometrically, the error comes from squares in a strip of fixed width near the circumference. Since the circumference is proportional to L , both the area of this strip and the number of squares in it are of order at most L .

We now return to the first quadrant. Let $A(L)$ be the number of lattice points (p, q) with $p, q > 0$ satisfying $p^2 + q^2 \leq L^2$. The non-axis points of the full disk are distributed equally among the four quadrants. The four half-axes contain a total of $4[L]$ nonzero lattice points, where $[L]$ denotes the greatest integer less than or equal to L .

Therefore, $B(L) = 1 + 4[L] + 4A(L)$, and hence $A(L) = \frac{B(L) - 1 - 4[L]}{4}$.

Using (1), and noting that the number of axis points also grows at most proportionally to L , we obtain:

$$A(L) = \frac{\pi L^2}{4} + O(L) \quad (2)$$

Thus, the leading term in the number of lattice points in the first-quadrant quarter-disk equals its area. The difference grows at most proportionally to L . This result concerns all lattice points, not only the coprime ones. A separate argument is needed to count the coprime points, whose number is denoted by $P(L)$.

To isolate the coprime points, we must remove the lattice points whose two coordinates share a prime divisor. Points whose coordinates are both divisible by d have the form $(p, q) = (da, db) = d(a, b)$.

Since $a^2 + b^2 \leq \left(\frac{L}{d}\right)^2$, the number of such points is $A(L/d)$. For example, subtracting $A(L/2)$ from $A(L)$ removes points obtained by doubling the coordinates of points in the quarter-disk of radius $L/2$. Subtracting $A(L/3)$ removes the points obtained by tripling the coordinates of points in the quarter-disk of radius $L/3$. However, if both $A(L/2)$ and $A(L/3)$ are subtracted, points obtained by multiplying coordinates by six are removed twice. The extra subtraction must therefore be corrected. This is the inclusion-exclusion principle. The successive subtractions and corrections needed to remove all non-coprime lattice points lead to the Möbius function, denoted by μ .

The Möbius function is defined as follows:

1. $\mu(1) = 1$.
2. If the square of a prime divides d , then $\mu(d) = 0$. For example, $\mu(12) = 0$ because 12 is divisible by $4 = 2^2$.
3. If d is a product of k distinct primes, then $\mu(d) = (-1)^k$. For example, $\mu(6) = (-1)^2 = 1$ because $6 = 2 \times 3$.

The essential property of this function is $\sum_{d|1} \mu(d) = \mu(1) = 1$, where $d | m$ means that d divides m . For $m > 1$, $\sum_{d|m} \mu(d) = 0$.

For example, $\sum_{d|6} \mu(d) = \mu(1) + \mu(2) + \mu(3) + \mu(6) = 1 - 1 - 1 + 1 = 0$.

Indeed, if the distinct prime divisors of m are $r_1, r_2, \dots, r_k$, then for $m > 1$ the sum $\sum_{d|m} \mu(d)$ equals the binomial expansion of $(1 - 1)^k$ and is therefore zero.

These facts and the role of the Möbius function give:

$$P(L) = \sum_{d \leq L} \mu(d) A\left(\frac{L}{d}\right) \quad (3)$$

In (3), the Möbius function carries out the inclusion-exclusion process. From (2):

$$A\left(\frac{L}{d}\right) = \frac{\pi L^2}{4d^2} + O\left(\frac{L}{d}\right) \quad (4)$$

Substituting (4) into (3) gives

$$P(L) = \frac{\pi L^2}{4} \sum_{d \leq L} \frac{\mu(d)}{d^2} + \sum_{d \leq L} \mu(d) O\left(\frac{L}{d}\right) \quad (5)$$

Three facts are now important.

1. It is known that $\sum_{d \leq L} 1/d$ is of order $\log L$. The magnitude of $\sum_{d \leq L} \mu(d)O(L/d)$ is bounded by $L \sum_{d \leq L} |\mu(d)|/d$, which is at most $L \sum_{d \leq L} 1/d$.

Therefore:

$$\sum_{d \leq L} \mu(d)O\left(\frac{L}{d}\right) = O(L \log L)$$

2. The series $\sum_{d=1}^{\infty} \mu(d)/d^2$ converges absolutely and equals $6/\pi^2$. If $\mathbb{P}$ denotes the set of primes, the Euler product and the classical value of the Riemann zeta function at two give [11], [12]:

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \prod_{r \in \mathbb{P}} \left(1 - \frac{1}{r^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \quad (6)$$

3. The infinite series can be divided into the two sums $\sum_{d \leq L} \mu(d)/d^2$ and $\sum_{d > L} \mu(d)/d^2$.

Moreover:

$$\left| \sum_{d > L} \frac{\mu(d)}{d^2} \right| \leq \sum_{d > L} \frac{1}{d^2} = O\left(\frac{1}{L}\right) \quad (7)$$

Combining these observations gives:

$$P(L) = \frac{\pi L^2}{4} \times \frac{6}{\pi^2} + O\left(\frac{1}{L}\right) + O(L \log L)$$

and therefore:

$$P(L) = \frac{3L^2}{2\pi} + O(L \log L) \quad (8)$$

For sufficiently large L , the number of paths of length at most L is therefore approximately $\frac{3L^2}{2\pi}$.

Equations (2) and (8) also give:

$$\lim_{L \rightarrow \infty} \frac{P(L)}{A(L)} = \lim_{L \rightarrow \infty} \frac{3L^2/(2\pi) + O(L \log L)}{\pi L^2/4 + O(L)} = \frac{6}{\pi^2} \approx 0.607927$$

Thus, about 60 percent of lattice points are coprime points.

To see how close the approximations are to the exact count, define:

$$P_1(L) = \sum_{d \leq L} \mu(d)A\left(\frac{L}{d}\right),$$

$$P_2(L) = \sum_{d \leq L} \mu(d) \frac{\pi L^2}{4d^2},$$

and

$$P_3(L) = \frac{3L^2}{2\pi}.$$

$P_1(L)$ is the exact value in (3). The value $P_2(L)$ results from replacing $A(L/d)$ in $P_1(L)$ by its approximation $\pi L^2/(4d^2)$. The value $P_3(L)$ is obtained by omitting the lower-order term $O(L \log L)$ from (8).

For example, when $L = 20$, direct counting gives $P(20) = 191$. In addition,

$$\begin{aligned} P_1(20) &= \mu(1)A(20) + \mu(2)A(10) + \dots + \mu(19)A\left(\frac{20}{19}\right) + \mu(20)A(1) \\ &= (1)(294) + (-1)(69) + (-1)(28) + (0)(15) + (-1)(8) \\ &\quad + (1)(6) + (-1)(4) + (0)(3) + (0)(1) + (1)(1) \\ &\quad + (-1)(1) + (0)(1) + (-1)(1) + (1)(1) + (1)(0) \\ &\quad + (0)(0) + (-1)(0) + (0)(0) + (-1)(0) + (0)(0) = 191. \end{aligned}$$

Likewise,

$$P_2(20) = \mu(1) \left(\frac{\pi}{4}\right) (20)^2 + \mu(2) \left(\frac{\pi}{4}\right) (10)^2 + \dots + \mu(19) \left(\frac{\pi}{4}\right) \left(\frac{20}{19}\right)^2 + \mu(20) \left(\frac{\pi}{4}\right) (1)^2$$

$$= 100\pi \left(1 - \frac{1}{4} - \frac{1}{9} + 0 - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + 0 + 0 + \frac{1}{100} - \frac{1}{121} + 0 - \frac{1}{169} + \frac{1}{196} + \frac{1}{225} + 0 - \frac{1}{289} + 0 - \frac{1}{361} + 0\right) \approx 190.19,$$

and

$$P_3(20) = \frac{3}{2\pi} (20)^2 = \frac{600}{\pi} \approx 190.99.$$

The approximate formulas give values very close to the exact count. Table I shows the same agreement for several values of L .

Table I) Comparison of exact and approximate counts of coprime points.

Method	$L = 10$	$L = 20$	$L = 40$	$L = 100$
Direct point count	47	191	765	4771
$P_1(L)$	47	191	765	4771
$P_2(L)$	48.40	190.19	765.54	4777.17
$P_3(L)$	47.75	190.99	763.94	4774.65

We conclude the solution with a brief examination of a rectangle in place of the unit square.

Suppose the path lies in a rectangle whose adjacent sides have lengths $a > 0$ and $b > 0$. The same unfolding associates a coprime pair (p, q) with a straight line of slope $\frac{bq}{ap}$.

In the unfolded plane, horizontal displacements of size ap and vertical displacements of size bq produce a segment from the origin to (ap, bq) . The condition that its length does not exceed L is $a^2p^2 + b^2q^2 \leq L^2$.

In the first quadrant, the region determined by this bound is an ellipse with semiaxes L/a and L/b . An argument similar to the one above shows that the number of simple paths of length at most L is asymptotic to $\frac{3L^2}{2\pi ab}$.

The leading expression depends only on the area ab of the rectangle. The individual side lengths appear only through their product.

5) Conclusion

The problem posed at the beginning of this article asked for the number of nonrepeating closed paths on an object when the path length is bounded.

If the meanings of *path*, *size*, and *object* are changed in a fundamental way, counting paths below a fixed length becomes a much deeper problem. One of Mirzakhani's results shows that, for a fixed hyperbolic surface X of genus g with n suitable boundary components or marked points, the number of simple closed curves of length at most L follows a power law [7], [8]. In compact form, the result is:

$$s_X(L) \sim c_X L^{6g-6+2n}$$

Here, $s_X(L)$ is the counting function, $6g - 6 + 2n$ is the effective dimension of the relevant space, and $c_X > 0$ is a constant that depends on the geometry of the surface. The precise meaning of these objects, the exact form of the constant, and the hypotheses of the theorem are given in the specialized references.

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